# *The Rankine Trochoidal Wave.*

By Sir George Greenhill, F.R.S.

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Rankine's Wave Theory of 1862 is given in the 'Philosophical Transactions,' vol. 153, p. 127, "The Exact Form of Waves at the Surface of Deep Water." The theory finds favour with the naval architect from its simple geometrical structure, intelligible to a student of elementary mathematics.

But this student does not like to be reminded that the waves at sea are not trochoidal, as this would require a supernatural state of internal commotion in the water, involving a distribution of molecular rotation. Sea waves are not permanent, but in a state of perpetual growth and decay, in a procession of group motion.

The Rankine theory will serve, however, as a first introduction to the structure of wave motion in water, so Rankine's treatment is reproduced here, with some simplification in his geometry. But, so far, the Rankine trochoidal wave has defied any effort of generalisation, as to a kindred state of wave motion in shallow water, or to the stationary wave in reflexion at a wall. The equation of continuity breaks down, and cannot be satisfied; cavitation takes place, or else overcrowding of the particles of water.

At the recent meeting of the Institution of Naval Architects, March, 1917, I have shown that, if thin parallel vertical walls are introduced into the water, perpendicular to the wave crest, without disturbing the wave motion, dividing up the water into compartments, these water compartments may be sheared over each other in an arbitrary manner, giving the wave front an échelon appearance, like a line of advancing infantry, and the wave motion will continue in each compartment.

The motion may also be sheared in these compartments by inclining the walls at the same angle with the vertical, like a row of books sloping over on a shelf. The circular orbit of a particle in the Rankine wave motion will then slope over at this angle, and his waves can exist between two parallel walls, either vertical or else sloping over at any assigned angle.

When the walls are brought close together and the shear steps are small, we can pass from discrete steps to continuous shear, uniform or variable; the wave front can then advance at any angle with the plane of the water motion, or in any curve.

With a straight crest, the vertical section of the surface by a plane

parallel to the shear is the original trochoid, and by a perpendicular plane is a sinusoid, by another plane a foreshortened trochoid.

These curves can be shown off as various aspects or projections of a revolving helix of wire, which ought to have its axis horizontal to imitate the wave motion. But it is more convenient in the experiment to let the helix hang vertically by a thread and revolve, and to treat the axis as horizontal in imagination.

Any projection by parallel lines of the helix on the floor will be a trochoid; a cycloid, if a line of projection can touch the helix.

A projection by horizontal lines on a vertical wall will be a sinusoid; any other projection in general will be an elongated or foreshortened trochoid.

In this way it is possible to break new ground in the Rankine trochoidal wave, and extend the treatment in some fresh directions. But here a delicate question arises, requiring examination. The continuity of the motion is preserved in the compartments when sheared, and also the free surface condition of constant pressure. But the continuity of the pressure in crossing a wall must be examined, as this may break down, and some new variable field of force must be introduced in addition to the constant field of vertical gravity, if the continuity of pressure is to be preserved when the walls or diaphragms are suppressed.

Reduced to a standing wave by a reversal of the wave velocity, $k$ or U, or else by taking a moving origin, as in a ship moving with the waves, the motion is treated as steady motion, and the equation of continuity and of pressure is somewhat easier to investigate. This has been carried out by Rankine in his paper, 'Phil. Trans.,' 1863, and we are at liberty to apply to this steady motion a constant, cross-current velocity, V, parallel to the wave crest. The effect is to make a stream line change from a trochoid to an elongated projection of the trochoid, on a vertical plane at an angle $\tan^{-1}(V/U)$ with the wave crest.

Advancing waves are called "rollers" and the standing wave may be called a "swell" (houle) in the language of the sailor. Any intermediate state between rollers and a swell will be encountered by a steamer moving straight across the crest.

To an observer on board the wave motion appears a swell if the steamer advances at the same rate as the rollers; and the relative motion appears steady, as in the trochoidal wave. But at any other velocity the motion appears to be a mixture of the roller and the swell.

The steamer may be moving faster or slower than the rollers, or in the opposite direction, head to sea. A sailing ship before the wind, and moving

through the water slower than the rollers, will be overtaken by the wave crest, and runs a risk of being "pooped."

When the course of the ship is parallel to the wave crest of the rollers, a "beam sea" is said to be encountered. The circular orbit of the Rankine roller is then drawn out into a helix in the motion relative to the ship. And this motion is added vectorially to the relative motion for a course perpendicular to the wave crest, when a course is steered at any other angle and given velocity. In this way the trochoidal motion of Rankine can be extended to motion in three dimensions.

Returning again to the rolling waves, and the circular orbit of a particle the effect of this cross current is to change the circle into a helix of stationary form advancing in rotation by passing through itself, as a screw through a nut or board, with no apparent alteration of the form of the wave surface.

The question arises further, whether it is legitimate to shear the Rankine motion in vertical planes parallel to the wave crest. In the previous states of shear the water was supposed divided up, by a number of thin vertical planes at close intervals, into compartments in which the motion can exist independently of the neighbours on each side, any pressure difference being taken up on the separating wall.

In the Rankine theory of the advancing waves, or rollers in the sailor's name, the orbit of a particle is a circle. But with respect to a vessel steaming with the waves, the wave motion will be stationary on the side relatively to the hull, while the water streams past, in trochoidal streamlines. The relative motion is reduced to a state of steady motion with respect to an origin on the vessel, as in a standing wave or swell. To an observer ashore the waves appear as rollers.

It is easier to examine the continuity and dynamical conditions in the steady motion of the swell, and so, following Rankine, a stream-line LBM is considered in fig. 1, a trochoid generated by a point B fixed on a circle, centre C, and circumference the wave-length $L = 2\pi R$, rolling at A on the underside of a horizontal line OA with the wave velocity $k$, and angular velocity (A.V.) $n = k/R = 2\pi k/L$.

The velocity of B is then $n \,.\, AB$ perpendicular to AB; and with $CB = r$, $FCB = \theta$, the co-ordinates of B are

$$x = R\theta + r\sin\theta, \qquad z = R + r\cos\theta, \qquad \theta = nt, \qquad R\theta = kt, \qquad (1)$$

and its component velocities

$$u = k + nr\cos\theta, \qquad w = -nr\sin\theta. \qquad (2)$$

Join FB, and draw the next trochoid $lbm$ a little lower down, produced by

rolling an equal circle, centre $c$ and $cb$ parallel to CB, on a line through $a$ at additional depth $\mathrm{A}a = \mathrm{C}c$.

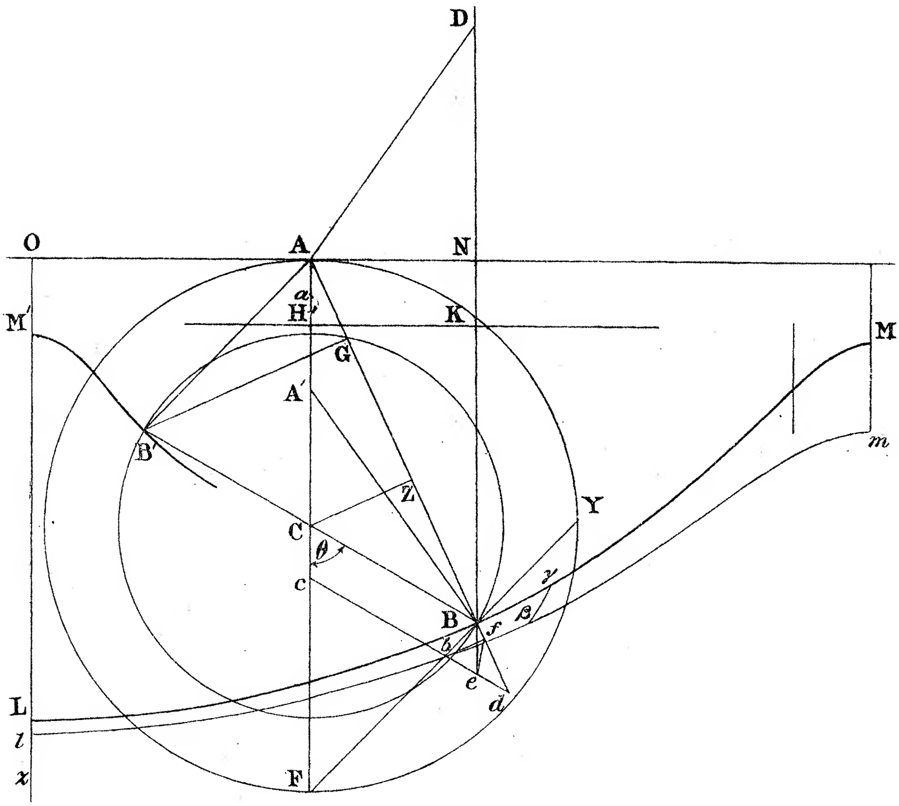


Fig. 1.

Then, $cb$ crossing AB, NB in $d$, $e$, the infinitesimal figure B$bedf$B is a small reduced copy of AB′CBGA; so that $be = ed = ef$, and $bfd$ is a right angle; and $bf$ may be taken as a tangent element of the trochoid through $b$.

Then

$$be/\mathrm{CB} = \mathrm{B}e/\mathrm{AC}: \tag{3}$$

and with $\mathrm{B}e = \mathrm{C}c = \mathrm{A}a = dc$, and $cb = r'$, $be = r - r' = -dr$,

$$(r-r')/r = -dr/r = dc/\mathrm{R}, \tag{4}$$

so that $r$ diminishes in G.P. with the depth $c$ increasing in A.P., and in accordance with the Exponential Theorem,

$$r = \mathrm{R}e^{-mc}, \qquad m\mathrm{R} = 1. \tag{5}$$

The flow across B$b$ due to the velocity $q = n\,.\,\mathrm{AB}$ perpendicular to AB is

$$n\,.\,\mathrm{AB}\,.\,\mathrm{B}f = n\,.\,\mathrm{AB}\,.\,\mathrm{AG}\,.\,\mathrm{B}e/\mathrm{A}c = n(\mathrm{R}^2 - r^2)(dc/\mathrm{R}), \tag{6}$$

the same for all sections such as B$b$, so that the space between the trochoids can flow full bore and the condition of continuity is assured.

The arc B$\gamma$ = AB . $d\theta$ for an advance $d\theta$ of the phase angle $\theta$; so that an element of area B$\gamma\beta b$ between the two trochoids is

$$\mathrm{B}\gamma \,.\, \mathrm{B}f = \mathrm{AB} \,.\, \mathrm{B}f \,.\, d\theta = \left(\mathrm{R} - \frac{r^2}{\mathrm{R}}\right) dc\, d\theta\,; \tag{7}$$

and the area of a half wave-length between the trochoids LBM, $lbm$ is

$$\pi\left(\mathrm{R} - \frac{r^2}{\mathrm{R}}\right) dc. \tag{8}$$

This is the area of a horizontal stratum of the same half wave-length $\frac{1}{2}\mathrm{L} = \pi\mathrm{R}$ and of depth $dc_0$, if

$$\pi\mathrm{R}\, dc_0 = \pi\left(\mathrm{R} - \frac{r^2}{\mathrm{R}}\right) dc, \tag{9}$$

$$\frac{dc_0}{dc} = 1 - \frac{r^2}{\mathrm{R}} = 1 - e^{-2mc}, \tag{10}$$

$$c_0 = c + \tfrac{1}{2}\mathrm{R}e^{-2mc} = c + (r^2/2\mathrm{R}), \tag{11}$$

and $c_0 - c = r^2/2\mathrm{R}$ is the rise in height of the central line through C of the trochoid LBM over the height when the stream LBM is flattened into a horizontal line in still water. The sea "rises" above its former mean level by this amount at the surface.

Thus $c_0 - c$ may be taken as the rise on the side of the mean water line of a vessel, several wave-lengths long, moored in the current of these standing trochoidal waves. And the flow between any two trochoid stream-lines a finite distance apart is $k$ times the depth of the stratum when at rest.

The curve B$b$ prolonged is shown in Froude's figure of Rankine's memoir, and is identified, by the property of the constancy of the subtangent CF, as a logarithmic curve, referred to oblique axes, one vertical, the other inclined at the angle $\theta$. This may be called the curve of a bulrush, waving in the water. The curve is recognised also in the trajectory of a sphere in still water or air when the resistance varies as the velocity; and then C descends with the terminal velocity $k$, and CB = $\mathrm{R}e^{-kt}$.

In accordance with roulette theory, the circle of inflexions of the trochoid is on the diameter AC, and the radius of curvature,

$$\rho = \mathrm{AB}^2/\mathrm{BZ}, \qquad q^2/g\rho = \mathrm{BZ}/\mathrm{AC}, \quad \text{with} \quad g/n^2 = \mathrm{AC}, \tag{12}$$

implying $k^2 = g\mathrm{R} = g\mathrm{L}/2\pi$,

$$(q^2/g\rho) + \cos\psi = \mathrm{BZ}/\mathrm{AC} + \mathrm{AZ}/\mathrm{AC} = \mathrm{AB}/\mathrm{AC}. \tag{13}$$

Then if $dp$ is the increase of pressure along B$f$ to the next trochoid

$$dp/\mathrm{B}f = \text{force along } \mathrm{B}f = w \,.\, \mathrm{AB}/\mathrm{AC}, \tag{14}$$

the same as in the circular orbit of the roller wave, with stationary centre C;

$$\frac{dp}{w}=\frac{\mathrm{AB}\,.\,\mathrm{B}f}{\mathrm{AC}}=\left(1-\frac{r^2}{p^2}\right)dc=dc_0, \tag{15}$$

so that the pressure gradient and the pressure are the same as in still water.

We are using Rankine's gravitation units here of the engineer; in the F.P.S. system (foot-pound-second) pressure $p$ is measured in lb/ft$^2$, density $w$ in lb/ft$^3$.

Take A′ the inverse point of A in the circle BC, and draw HK horizontal, bisecting AA′ at right angles. With BD = 2BK, the triangles ABC, BA′C, ADB, are similar

$$\mathrm{AC}/\mathrm{AB}=\mathrm{CB}/\mathrm{A'B}=\mathrm{AB}/\mathrm{BD},\quad \mathrm{AB}^2=\mathrm{AC}\,.\,\mathrm{BD}=2\mathrm{AC}\,.\,\mathrm{BK}, \tag{16}$$

$$q^2=n^2\,.\,\mathrm{AB}^2=g\,.\,\mathrm{AB}^2/\mathrm{AC}=2g\,.\,\mathrm{BK}, \tag{17}$$

so that the particle B moves along the trochoid as if on a smooth switchback railway, with velocity due to the level of HK.

In the liquid stream between LBM and *lbm*, the thickness is inversely as the velocity, $n$ . AB.

If R is the pressure at B of the particle, of weight W,

$$\mathrm{R}/\mathrm{W}=\cos\psi-q^2/g\rho=\mathrm{AZ}/\mathrm{AC}-\mathrm{BZ}/\mathrm{AC}=\mathrm{AB}/\mathrm{AC}, \tag{18}$$

so that the pressure per unit length of the stream

$$dp=\mathrm{R}/\mathrm{B}\gamma=\mathrm{W}/\mathrm{B}\gamma(\mathrm{AB}/\mathrm{AC})=w\,.\,\mathrm{B}f(\mathrm{AB}/\mathrm{AC}), \tag{19}$$

$$\frac{dp}{w}=\frac{\mathrm{AC}^2-\mathrm{BC}^2}{\mathrm{AC}^2}\mathrm{BC}=\left(1-\frac{r^2}{\mathrm{R}^2}\right)dc=dc_0, \tag{20}$$

as before.

The liquid particles are crowded together, and bunch up, like traffic on the road, near L, where the speed is small, and then tail out towards M as the speed increases.

Treating B$\beta$ as a horizontal step $dx$ from B to the next trochoid at $\beta$, the difference of pressure at B and $\beta$ is

$$dp=w\,.\,dc_0=w\,.\,\mathrm{AB}\,.\,\mathrm{B}f/\mathrm{AC}, \tag{21}$$

$$dx=\mathrm{B}\beta=\mathrm{B}f/\sin\psi, \tag{22}$$

$$\frac{dp}{w\,dx}=\frac{\mathrm{AB}\sin\psi}{\mathrm{AC}}=\frac{\mathrm{AN}}{\mathrm{AC}}=\frac{r}{\mathrm{R}}\sin\theta=e^{-mc}\sin\theta \tag{23}$$

With thin compartments of breadth $dy$, sheared at angle S, tan S = $dx/dy$, and $dp$ the pressure difference on the sides of a wall,

$$dp/w\,dy=dp/w\,dx\,.\,\tan\beta=e^{-mc}\sin\theta\tan\mathrm{S}, \tag{24}$$

requiring the introduction of a field of force of this intensity perpendicular to the wall.

The échelon waves cannot exist, then, with the wave front slewed through this angle S without the introduction of this horizontal field of force perpendicular to the vertical plane of the circular orbits or trochoidal stream-lines. So also a vertical shear of the compartments would require an additional field of

$$\mathrm{Y} = [1+(r/\mathrm{R})\cos\theta]\tan \mathrm{S}. \tag{25}$$

But if the walls of the compartments lean over at an angle $\alpha$ with the vertical, the trochoidal stream-lines can still remain comparable with each other in the steady motion of the standing wave, and in the rollers the circular orbits are inclined at $\alpha$ to the vertical.

This motion can persist when the walls are removed, without a need of an additional field of force, provided $g$ is replaced by $g\cos\alpha$.

In the associated state of wave motion of small displacement, the velocity function $\phi$ is changed from

$$\phi = \mathrm{A}e^{-mz}\cos(mx+nt) \quad \text{to} \quad \mathrm{A}e^{-mz\cos\alpha-my\sin\alpha}\cos(mx+nt), \tag{26}$$

with wave crest perpendicular to a wall sloping at angle $\alpha$, and with circular orbits parallel to the wall.

So it is worth while examining if it is possible to cut down the original free surface of the Rankine wave motion between the slanting walls, to a mean slope such that it can still remain a surface of constant pressure, without interfering with the continuity of the pressure in the interior.

Consider the liquid motion between two vertical walls parallel to the wave crest of the standing trochoidal waves in steady streaming motion, and the trochoidal path of a water particle when a horizontal field of gravity, $g\tan\alpha$, is introduced perpendicular to the walls in addition to the vertical field $g$, which we assume will not interfere with the motion.

Cover the free surface with a fixed cylinder moulded to the surface of the running water in stationary waves, capable of taking a variation of pressure.

The liquid motion is altered by the new field, $g\tan\alpha$, and continuity will be preserved, but a surface of equal pressure will change in the interior of the water; and if a new surface of equal pressure is composed of trochoidal stream-lines, the water above it may be removed, and a new state of motion is obtained in a field of gravity $g\sec\alpha$, and the plane of a trochoidal path is inclined at an angle $\alpha$ with the resultant gravity.

Suppose, then, the original $g$ is reduced to $g\cos\alpha$, and a new field $g\sin\alpha$ is introduced perpendicular to the walls; this is equivalent to a field $g$ at an angle $\alpha$ with the walls and Bernoulli's equation in (40) changes with absolute units into

$$\frac{p}{\rho} - gz\cos\alpha + gy\sin\alpha + \tfrac{1}{2}q^2 = g\mathrm{H}, \tag{27}$$

where H is constant along a stream-line;

$$\frac{p}{\rho} - gc\cos\alpha - gr\cos\alpha\cos\theta + gy\sin\alpha + \tfrac{1}{2}n^2(\mathrm{R}^2 + 2\mathrm{R}r\cos\theta + r^2) = \mathrm{H}. \quad (28)$$

Then $p$ is constant along the stream-line, if $n^2\mathrm{R} = g\cos\alpha$; and then

$$y\sin\alpha - c\cos\alpha + \tfrac{1}{2}\frac{r^2}{\mathrm{R}}\cos\alpha = \text{a constant} \quad (29)$$

along the profile of a vertical section parallel to the wave crest and perpendicular to the walls, a line connecting points of equal pressure.

A surface of equal pressure is built up of these profiles, and since $r = \mathrm{R}e^{-mc}$, $m\mathrm{R} = 1$, to satisfy the continuity, the profile is the exponential curve

$$y\tan\alpha - c + \tfrac{1}{2}\mathrm{R}e^{-2mc} = \text{a constant}, \quad (30)$$

having an asymptote parallel to $y\tan\alpha = c$, the new horizontal.

Thus the water may be drawn off above one of these surfaces of equal pressure, and, by tilting the walls to an angle $\alpha$ with the vertical, the new field of $g$ may be made vertical. A new system of stationary trochoidal waves is then realised in which the plane of a trochoidal stream-line (or of the circular orbit of the rollers) is parallel to the sloping walls, the wave amplitude dying out in G.P. as it recedes from the upper side of the sloping wall.

The waves diminish in height towards the under side of a sloping wall, and so illustrate the form of a wave under the sloping bow of a steamer.

These new stationary trochoidal waves may be changed into moving rollers by reversing the general stream of the water in the steady motion, and then the velocity of advance will be

$$k' = k\sqrt{(\cos\alpha)} = \sqrt{(g\mathrm{R}\cos\alpha)} = \sqrt{\left(\frac{g\mathrm{L}}{2\pi}\cos\alpha\right)}. \quad (31)$$

The curve on the wall or sloping shore of the surface of the water will be a trochoid, stationary, or advancing with velocity $k'$, the wave crests reaching out to the offing at sea with diminishing amplitude, and tending to the small motion considered previously in (26) of waves along a sloping beach.

The analytical expression of the Rankine moving wave or roller may be stated and generalised, in the Lagrangian form, with O$x$ in the surface perpendicular to the wave crest, and O$z$ vertically downward,

$$x = a + r\sin\theta, \qquad z = c + s\cos\theta, \qquad \theta = ma + nt, \quad (32)$$

where $r$, $s$ are supposed functions of the depth $c$, determined from the condition of the equation of continuity. Afterwards the pressure equation can be investigated, and the condition for a free surface, if possible.

Then by differentiation

$$dx/da = 1+mr\cos\theta, \qquad dz/da = -ms\sin\theta, \tag{33}$$

$$dx/dc = (dr/dc)\sin\theta \qquad dz/dc = 1+(ds/dc)\cos\theta, \tag{34}$$

$$\frac{d(x,z)}{d(a,c)} = (1+mr\cos\theta)\left(1+\frac{ds}{dc}\cos\theta\right)+ms\frac{dr}{dc}\sin^2\theta$$

$$= 1+\left(mr+\frac{ds}{dc}\right)\cos\theta+mr\frac{ds}{dc}\cos^2\theta+ms\frac{dr}{dc}\sin^2\theta, \tag{35}$$

and this must be independent of $\theta$ for Lagrange's equation of continuity to be satisfied. Thus

$$r\,ds/dc = s\,dr/dc, \quad 1/r\,.\,dr/dc-1/s\,.\,ds/dc = 0, \quad r/s = e, \text{ a constant}, \tag{36}$$

$$mr+ds/dc = 0, \qquad 1/s\,.\,ds/dc = -em, \qquad s = \mathrm{R}e^{-emc}, \tag{37}$$

and the particle orbit is an ellipse.

In the associated steady motion swell

$$u = \mathrm{U}+nr\cos\theta, \qquad w = -ns\sin\theta, \qquad \theta = ma, \tag{38}$$

$$q^2 = u^2+w^2 = \mathrm{U}^2+2\mathrm{U}nr\cos\theta+n^2(r^2\cos^2\theta+s^2\sin^2\theta) \tag{39}$$

and in Bernoulli's equation and a gravity field

$$\mathrm{H} = p/w-z+q^2/2g \tag{40}$$

is constant along a stream-line. Here all the terms are of the dimensions of a length.

But this condition cannot be satisfied unless $r = s$, $e = 1$, and the orbit of the roller a circle, and we arrive at the Rankine motion, and then

$$\mathrm{H} = p/w-c-r\cos\theta+\mathrm{U}^2/2g+\mathrm{U}nr/g\,.\,\cos\theta+n^2r^2/2g, \tag{41}$$

is constant along a stream-line, if

$$\mathrm{U}n = g, \qquad n = \mathrm{U}m, \qquad \mathrm{U}^2 = gm = g\mathrm{L}/2\pi. \tag{42}$$

Otherwise, if $r$ and $s$ are unequal, an additional field of force must be introduced to make a free surface possible.

Reducing Rankine's wave to a stationary wave by a combination with a reflected wave

$$x = a+r\sin ma\cos nt, \quad z = c+r\cos ma\cos nt, \quad dr/dc = -mr, \tag{43}$$

and a liquid particle oscillates in a straight line; but

$$d(x,z)/d(a,c) = 1-m^2r^2\cos^2 nt, \tag{44}$$

and the equation of continuity is not satisfied; so it is useless to proceed further, unless $r^2$ is treated as negligible.

So also, if we tried to generalise the equations of wave motion of small displacement in shallow water of depth $h$, with

$$x = a+\mathrm{A}\,\mathrm{ch}\, m(h-c)\sin(ma+nt), \quad z = c+\mathrm{B}\,\mathrm{sh}\, m(h-c)\cos(ma+nt), \quad (45)$$

the equation of continuity cannot be satisfied, unless we suppose $\mathrm{A} = \mathrm{B}$, and AB insensible, and then we arrive at the ordinary theory for waves of small displacement in shallow water, and exact equations cannot be maintained.

In the usual theory of hydrodynamics of the straight crested wave of small displacement in a canal, where the velocity function $\phi = \mathrm{P}\cos(mx+nt)$ and P is a function of $y$ and $z$, satisfying the condition of continuity

$$(d^2\mathrm{P}/dy^2)+(d^2\mathrm{P}/dz^2) = m^2\mathrm{P}, \quad (46)$$

and the conditions at the boundary. The lines of flow are given by

$$dx/\mathrm{P} = dy/\mathrm{Q} = dr/\mathrm{R}, \qquad \mathrm{P, Q, R} = d\phi/dx,\ d\phi/dy,\ d\phi/dz, \quad (47)$$

and if two solutions can be obtained of these differential equations, in the form $u = a$, $v = b$, then $\mathrm{F}(u, v) = 0$ will give a bounding surface which can hold the motion, and this is the solution of the partial differential equation

$$\mathrm{P}(dr/dx)+\mathrm{Q}(dr/dy) = \mathrm{R}. \quad (48)$$

Any of the solutions obtained hitherto in hydrodynamics for wave motion can be sheared continuously in a vertical plane perpendicular to the straight wave crest at an angle S, constant or variable, and the wave crest may be made to take any curve, and the shape of the containing vessel may be determined to hold the new motion.

But if continuity in the pressure is to be preserved, a field of horizontal force must be introduced

$$\mathrm{Y} = \frac{1}{w}\frac{dp}{dy} = \frac{1}{w}\frac{dp}{dx}\tan \mathrm{S}, \quad (49)$$

and in the pressure equation, in gravitation units,

$$\frac{p}{w} - z + \frac{d\phi}{gdt} = \text{constant}, \quad (50)$$

so that in the shear, with $z$ unaltered,

$$\frac{1}{w}\frac{dp}{dx} = -\frac{d^2\phi}{gdx\,dt} = \frac{mn}{g}\phi. \quad (51)$$

### *Curl of a Vector and Molecular Rotation, Circulation, and Rotation.*

Interpreted in uniplanar motion and a vector velocity $q$, with components $u$, $v$, the circulation is defined as the line integral of the tangential component of the velocity; divided by twice the area, this is called the curl or rotation, by analogy with a rigid body, and of a liquid moving bodily as if solid.

Thus if $l$, $m$ denote the direction cosines of the outward drawn normal of a closed curve $l = dy/ds$, $m = -dx/ds$, and the circulation

$$C = \int (lv - mu)\, ds \tag{52}$$

while twice the area

$$2A = \int (lx - my)\, ds. \tag{53}$$

Draw the line $NP_1P_2P_3P_4 \ldots$ in fig. 2 parallel to $Ox$, entering the closed

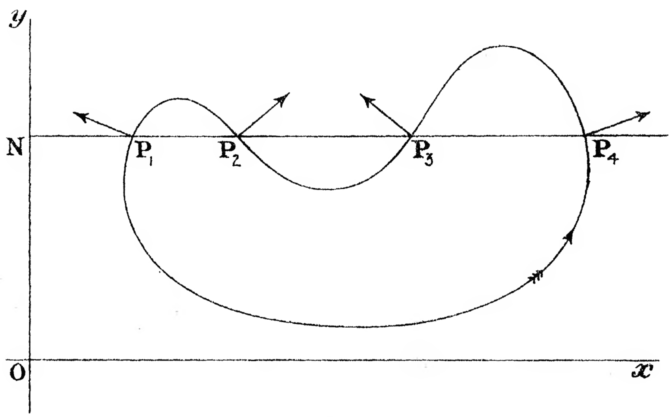


Fig. 2.

curve at $P_1$, $P_3$, . . . and leaving it at $P_2$, $P_4$, . . . ; then

$$lv\, ds = -v dy \text{ at entry}, \qquad +v dy \text{ at exit},$$

$$= \int \frac{dv}{dx}\, dx\, dy. \tag{54}$$

and similarly

$$mu\, ds = \int \frac{du}{dy}\, dx\, dy; \tag{55}$$

so that

$$C = \iint \left(\frac{dv}{dx} - \frac{du}{dy}\right) dx\, dy, \qquad A = \iint dx\, dy. \tag{56}$$

For a small circuit of an area element $dx\, dy$, enclosing a point P, the rotation becomes

$$\tfrac{1}{2}(dv/dx - du/dy). \tag{57}$$

For a bodily rotation about O, with A.V. $\omega$, $\quad v = \omega x$, $\quad u = -\omega y$,

$$C = \omega \int (lx - my)\, ds = 2\omega A. \tag{58}$$

Consider the circulation in the trochoidal motion round the element of area $Bb\beta\gamma$ in fig. 1.

The circulation over $B\gamma$ is

$$n \,.\, AB \,.\, B\gamma = n \,.\, AB^2 \,.\, d\theta = n(R^2 - 2Rr\cos\theta + r^2)\, d\theta \tag{59}$$

and the counter circulation over $b\beta$ is

$$n(\mathrm{R}^2 - 2\mathrm{R}r' \cos\theta + r'^2)\, d\theta, \tag{60}$$

the difference being

$$-2n\mathrm{R}(r-r')\cos\theta\, d\theta + n(r^2 - r'^2)\, d\theta,$$

$$= -2nr\cos\theta\, dc\, d\theta + n(r+r')\frac{r}{\mathrm{R}}\, dc\, d\theta,$$

$$= -2nrd(\sin\theta)\, dc + n(r+r')\frac{r}{\mathrm{R}} dc\, d\theta. \tag{61}$$

The counter circulation over $b\mathrm{B}$ is

$$n \,.\, \mathrm{AY} \,.\, b\mathrm{B} = n \,.\, \mathrm{AB} \,.\, bf = n \,.\, \mathrm{AB} \,.\, \mathrm{GB}' \,.\, \mathrm{B}e/\mathrm{AC},$$

$$= 2n \,.\, \Delta\, \mathrm{ABB}'\, dc/\mathrm{R} = 2n \,.\, \Delta\, \mathrm{ABF}\, dc/\mathrm{R}$$

$$= 2n\mathrm{R}r \sin\theta\, dc/\mathrm{R} = 2nr\sin\theta\, dc, \tag{62}$$

and the circulation over $b\mathrm{B}$ and $\beta\gamma$ is

$$2nr\, d(\sin\theta)\, dc. \tag{63}$$

Thus the total circulation round $\mathrm{B}\gamma\beta b$ is

$$n(r+r')r/\mathrm{R} \,.\, dc\, d\theta. \tag{64}$$

Dividing by twice the area, the rotation

$$\omega = n\frac{(r+r')r/\mathrm{R} \,.\, dc\, d\theta}{2(\mathrm{R} - r^2/\mathrm{R})\, dc\, d\theta} = \frac{n(r+r')r}{2(\mathrm{R}^2 - r^2)} \to \frac{nr^2}{\mathrm{R}^2 - r^2}. \tag{65}$$

Such liquid molecular rotation could not have been set up from rest in a perfect fluid under the action of natural forces and the pressure of the surrounding liquid, on the principle of normality. A cylindrical or spherical element of the liquid for instance, if solidified, could not acquire a rotation, or lose it. As stated by Aristotle in 'De Cœlo,' *ἥκιστα δὲ κινητικὸν ἡ σφαῖρα διὰ τὸ μηδὲν ἔχειν ὄργανον πρὸς τὴν κίνησιν.*